Lecture 14: Momentum and momentum conservation

Last lecture we defined the momentum, $\vec{p} = m\vec{v}$, and the impulse $\vec{I} = \vec{F}\Delta t$. If the force is a constant we have,

$$\vec{I} = \vec{F}\Delta t = \Delta\vec{p} = \vec{p}_f - \vec{p}_i \quad (Constant\ force). \tag{1}$$

If the force varies with time, then we have a sum of force terms, which however can be replaced by an average force, so that,

$$\vec{I} = \sum_i \vec{F}_i \Delta t_i \to \vec{F}_{av}\Delta t. \tag{2}$$

Impulse example: Consider a car travelling at $60mph$ in a head on collision with a tree. The duration of the impact is $0.010ms$ after which the car comes to rest. Find the average force experienced by the driver during the impact. The driver weighs $180lb$.

Solution: We use the equation

$$\vec{F}_{av}\Delta = \vec{p}_f - \vec{p}_i. \tag{3}$$

The final momentum of the car is $\vec{p}_f = 0$. The fact that the collision is head on implies that we can assume that the motion is in one direction which we take to be the x-direction. We convert the initial speed to meters per second to find $v_i = 60/2.24m/s$, and we convert the drivers mass to kg, so that $m = 180/2.20kg$. The initial momentum of the driver is then,

$$p_i = mv_i = 2192kgm/s \tag{4}$$

The average force is then,

$$F_{av} = \frac{\Delta p}{\Delta t} = \frac{-p_i}{\Delta t} = \frac{-2192}{0.01} = -2.19 * 10^5 N \tag{5}$$

The minus sign indicates that the force of the tree on the car is in the negative x-direction. If we assume that the area of the human body is about $0.5m^2$, then the pressure ($= F_{av}/Area$) experienced by the driver is on average,

$$Average\ pressure = 4.38 * 10^5 N/m^2 \tag{6}$$

This is more than the human body can withstand, so that a head on collision at $60mph$ is usually fatal.

Lets figure out some of the other mechanical properties of the collision. If the mass of the car is $M = 1000kg$, then the force exerted on the tree during the collision is given by,

$$F_{tree} = \frac{(M+m)v_i}{\Delta t} = 2.9 * 10^6 N \tag{7}$$

and the pressure on the tree, assuming a tree of radius $20cm$ is

$$Average\ pressure\ on\ tree = \frac{F_{tree}}{0.4m} = 7.2 * 10^6 N/m^2 \tag{8}$$

This pressure is high enough to break many types of tree, so that it is not unusual to see a tree felled due to a high speed head on collision. Some of the other quantities of interest are as follows: The average power dissipated during the collision is $(M + m)v_i^2/(2\Delta t)$, the average deceleration is $F_{av}/m = 2700m/s^2 >> g$, the external work done by the tree is $W_e = (M + m)v_i^2/2$.

A special case: No external force, Collisions

A special case of the formula $\vec{F}\Delta t = \Delta\vec{p}$ is when there is no external force. The trick is to pick a problem where the external forces do not affect the motion. A really important example is the case of collisions. We shall only study collisions in one dimension. Consider two masses, m_1 and m_2 which have initial velocities v_{1i} and v_{2i}. After they collide, their final velocities are v_{1f}, v_{2f}. Assuming that no external forces act on the two masses during the collision, then we have,

$$\Delta\vec{p} = p_{1f} + p_{2f} - (p_{1i} + p_{2i}) = 0. \tag{9}$$

or,

$$m_1 v_{1f} + m_2 v_{2f} = m_1 v_{1i} + m_2 v_{2i} \tag{10}$$

Note that we have dropped the vector notation because we are treating a collision in one dimension. In writing down this equation we have chosen our "system" to be the two masses. The forces which occur during their collision are then "internal" forces and do not influence the total momentum of the system we have chosen. There are still external forces on the system: Gravity, the normal force and friction. However for the moment we assume that the motion is in the x-direction and the motion is frictionless so none of these forces changes the motion in the x-direction.

Given the initial velocities and the two masses, we would like to know the final velocities. However there are two unknown final velocities so we do not have enough information to solve the problem. To solve the problem we need to provide information about the energies involved in the collision. Now we consider two limiting cases: Elastic collisions where energy is conserved and; Perfectly inelastic collisions where the masses stick together after the collision.

Perfectly inelastic collisions

In a perfectly inelastic collision, the two masses stick together, in which case $v_{1f} = v_{2f} = v_f$. Then Eq. (10) can be solved to find that,

$$v_f = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2} \tag{11}$$

How much energy is lost in this collision? This is found from,

$$\Delta KE = KE_f - KE_i = \frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2 - \frac{1}{2}(m_1 + m_2)v_f^2 \tag{12}$$

For example consider two masses $m_1 = m_2 = 1kg$, with $v_{2i} = 0$ and $v_{1i} = 10m/s$. Then we find that $v_f = 5m/s$ and $KE_i = 50J$, $KE_f = 25J$, so that $\Delta KE = -25J$, which is the energy dissipated in the collision.

Elastic collisions

There is no loss of energy in an elastic collision, so that $KE_f = KE_i$, we then have,

$$\frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2 = \frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2 \tag{13}$$

We need to solve this equation in combination with the momentum conservation equation (Eq. (10)) in order to find the final velocities for elastic collisions. These two equations look pretty horrible to solve, however it turns out to be quite easy. First we rearrange Eq. (13) to find,

$$m_1(v_{1i}^2 - v_{1f}^2) = m_2(v_{2f}^2 - v_{2i}^2) \tag{14}$$

This can be factored to find,

$$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i}) \tag{15}$$

We may also rearrage the momentum conservation equation (Eq. (10)) as,

$$m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i}) \tag{16}$$

Using this in Eq. (15), we find that Eq. (15) reduces to,

$$v_{1i} + v_{1f} = v_{2i} + v_{2f} \tag{17}$$

This is a much simpler equation, which can be combined with the momentum convervation equation to find a complete solution, though it is a little messy.

Elastic collision examples

Consider two masses $m_1 = m_2 = 1kg$, $v_{2i} = 0$, $v_{1i} = 10m/s$, which undergo a head on elastic collision. Find the final velocities v_{1f} and v_{2f} of the two masses. From Eq. (17), we have,

$$v_{1f} = v_{2f} - 10 \tag{18}$$

Substitution into the momentum conservation equation yields,

$$20 - v_{2f} = -v_{2f} \tag{19}$$

Which shows that $v_{2f} = 10m/s$, and hence that $v_{1f} = 0$! The momentum and kinetic energy is transferred completely from mass one to mass two What happens if $m_1 >> m_2$? What happens if $m_1 << m_2$? What happens if $m_1 = m_2$, but $v_{2i} \neq 0$.